SECTION 14.5: CURVATURE AND NORMAL VECTORS

NORMAL VECTORS:

RECALL: If $\|\vec{r}(t)\|$ is constant, then $\vec{r}(t) \perp \vec{r}\,'(t)$. Since $\hat{T}(t)$ is a unit vector, $\|\hat{T}(t)\| = 1$ so $\hat{T}(t) \perp \hat{T}\,'(t)$ ...

DEFINITION: If $\hat{T}(t)$ is smooth, the **principal unit normal vector** is $\hat{N}(t) = \dfrac{\hat{T}\,'(t)}{\|\hat{T}\,'(t)\|}$

NOTE: If $\vec{r}(t)$ is the position of an object, $\hat{N}(t)$ is the direction the object is **turning**. Can you see why?

EXAMPLE 1: Find $\hat{T}(t)$ and $\hat{N}(t)$ for $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$ and verify $\hat{T}(t) \perp \hat{N}(t)$.

Ans: $\hat{T}(t) = \left\langle -\dfrac{5}{13}\sin(5t), \dfrac{5}{13}\cos(5t), \dfrac{12}{13} \right\rangle$, $\hat{N}(t) = \langle -\cos(5t), -\sin(5t), 0 \rangle$

EXAMPLE 2: Find $\hat{T}(t)$ and $\hat{N}(t)$ for $\vec{r}(t) = \left\langle t^2, t \right\rangle$ and verify $\hat{T}(t) \perp \hat{N}(t)$.

NOTE: In 2D, if $\hat{T}(t) = \langle f(t), g(t) \rangle$ then $\hat{N}(t) = \langle -g(t), f(t) \rangle$ or $\hat{N}(t) = \langle g(t), -f(t) \rangle$. (Do you see why?)
You can then select whichever vector for $\hat{N}$ that points towards the concave side of the curve ...

Ans: $\hat{T}(t) = \left\langle \dfrac{2t}{\sqrt{4t^2+1}}, \dfrac{1}{\sqrt{4t^2+1}} \right\rangle$, $\hat{N}(t) = \left\langle \dfrac{1}{\sqrt{4t^2+1}}, -\dfrac{2t}{\sqrt{4t^2+1}} \right\rangle$

DEFINITION: If $\hat{T}$ is smooth, the **binormal vector** is $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$

QUESTION: Can you show $\hat{B}(t)$ is a unit vector?

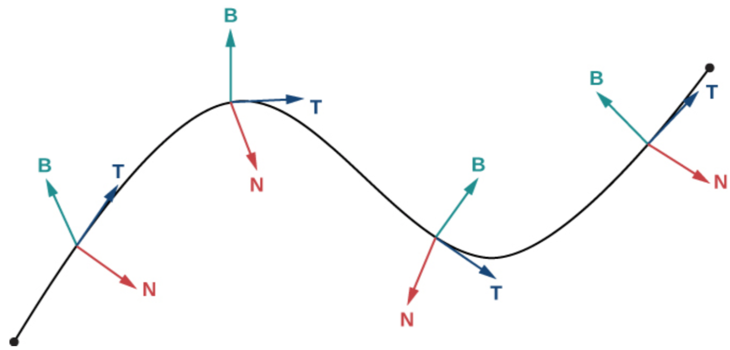
HINT: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\,\|\vec{w}\| \sin(\theta)$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

EXAMPLE 3: Find $\hat{B}(t)$ for $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$.

Verify $\hat{B}(t) \cdot \hat{T}(t) = 0$, $\hat{B}(t) \cdot \hat{N}(t) = 0$, and $\|\hat{B}(t)\| = 1$.

Ans: $\hat{B}(t) = \left\langle \dfrac{12}{13}\sin(5t), -\dfrac{12}{13}\cos(5t), \dfrac{5}{13} \right\rangle$

TNB FRAME: If $\vec{r}(t)$ is the position of an object, $\hat{T}(t)$, $\hat{N}(t)$ and $\hat{B}(t)$ comprise the **TNB** or **Frenet frame**:



A localized 3D coordinate system.

- The **osculating plane** is the plane determined by $\hat{T}(t)$ and $\hat{N}(t)$ with normal vector $\hat{B}(t)$.

  NOTE: We'll see shortly that this is the plane of motion.

- The **normal plane** is the plane determined by $\hat{N}(t)$ and $\hat{B}(t)$ with normal vector $\hat{T}(t)$.

- The **rectifying plane** is the plane determined by $\hat{T}(t)$ and $\hat{B}(t)$ with normal vector $\hat{N}(t)$.

EXAMPLE 4: Find the equation of the osculating plane to $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$ when $t = 0$.

Ans: $5z - 12y = 0$ or $z = \dfrac{12}{5}y$.

RECALL: If $\vec{r}$ is smooth, then $\hat{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ so that $\vec{v} = \|\vec{v}(t)\|\,\hat{T}(t)$.

THEOREM: If $\vec{r}(t)$ and $\hat{T}(t)$ are smooth, then there are functions $a_T(t)$ and $a_N(t)$ so that:

$$\vec{a}(t) = a_T(t)\hat{T}(t) + a_N(t)\hat{N}(t).$$

PROOF:

$$\begin{aligned}
\vec{a}(t) &= D_t[\vec{v}(t)] \\
&= D_t[\|\vec{v}(t)\|\,\hat{T}(t)] \\
&= D_t[\|\vec{v}(t)\|]\,\hat{T}(t) + \|\vec{v}(t)\| D_t[\hat{T}(t)] \\
&= D_t\left[\|\vec{v}(t)\|\right]\,\hat{T}(t) + \|\vec{v}(t)\|\,\hat{T}\,'(t) \\
&= D_t\left[\|\vec{v}(t)\|\right]\,\hat{T}(t) + \|\vec{v}(t)\|\,\|\hat{T}\,'(t)\|\hat{N}(t).
\end{aligned}$$

We define $a_T(t) = D_t\left[\|\vec{v}(t)\|\right]$ and $a_N(t) = \|\vec{v}(t)\|\,\|\hat{T}\,'(t)\|$, which proves the claim.

NOTE:

- The vector $a_T(t)\hat{T}(t)$ is the **tangential component of the acceleration**; $a_T(t)$ describes how much acceleration is in the direction you're traveling.
- The vector $a_N(t)\hat{N}(t)$ is the **normal vector component of acceleration**; $a_N(t)$ describes how much of the acceleration is in the direction you're turning. Note that $a_N(t) \geq 0$.
- We now know the acceleration (hence force) vector lives entirely in the osculating plane.

ALTERNATE FORMULAS FOR a_T:

$$a_T = \vec{a} \cdot \hat{T} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|} = \frac{d^2 s}{dt^2}$$

ALTERNATE FORMULAS FOR a_N:

$$a_N = \vec{a} \cdot \hat{N} = \|\vec{a} \times \hat{T}\| = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \sqrt{\|\vec{a}\|^2 - a_T^2}$$

EXAMPLE 5: For each of the v.v.f.'s below, find a_T and a_N and verify $\vec{a}(t) = a_T(t)\hat{T}(t) + a_N(t)\hat{N}(t)$.

NOTE: Refer to previous examples for $\hat{T}$ and $\hat{N}$ for each of the curves below.

1. $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$.

Ans: $a_T = 0$ and $a_N = 25$

2. $\vec{r}(t) = \left\langle t^2, t \right\rangle$.

Ans: $a_T = \dfrac{4t}{\sqrt{4t^2+1}}$ and $a_N = \dfrac{2}{\sqrt{4t^2+1}}$

CURVATURE:

RECALL: Given a smooth v.v.f. $\vec{r}(t)$, $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ is the principal unit tangent vector.

If $\vec{r}(t)$ is the position of an object, then $\hat{T}(t)$ is the direction the object is heading.

DEFINITION: Given a v.v.f. $\vec{r}$ where $\hat{T}$ is smooth, the **curvature**, κ is given by: $\kappa = \left\| \dfrac{d\hat{T}}{ds} \right\| = \|\hat{T}\,'(s)\|$

QUESTION: κ is the magnitude of the rate of change of _____________ with respect to _____________.

EXAMPLE 6: Calculate the curvature of $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$ using the definition.

Ans: $\kappa = \dfrac{25}{169}$

THOREM: Other formulas for curvature:

$$\kappa = \left\| \frac{d\hat{T}}{ds} \right\| = \frac{\|\hat{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\hat{T}\,'(t)\|}{\|\vec{v}(t)\|} \qquad \text{and} \qquad \kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

COMPONENTS OF ACCELERATION (REPRISE):

$$\vec{a}(t) = \frac{d^2 s}{dt^2}\,\hat{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^2 \hat{N}(t).$$

NOTE: There are two quadratic ideas happening here: a second derivative, d^2s/dt^2 in the direction of the motion and the square of the first derivative, $(ds/dt)^2$ in the normal direction. Coincidence?

EXAMPLE 7: Re-calculate the curvature of $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$ using the two formulas above.

Ans: $\kappa = \dfrac{25}{169}$

EXAMPLE 8: If $y = f(x)$ is smooth, we may parametrize the graph of f as $\vec{r}(t) = \langle t, f(t) \rangle$. Derive the formula:

$$\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

CIRCLE OF CURVATURE:

With normal vectors, we can now discuss more what curvature means.

DEFINITION:

- The **radius of curvature** at $\vec{r}(a)$ is $r_\kappa(a) = \dfrac{1}{\kappa(a)}$.
- The **center of curvature** is the terminal point of the vector $r_\kappa(a)\,\hat{N}(a)$ with initial point $\vec{r}(a)$.
- The **circle of curvature** is the circle defined by the center radius of curvature.

EXAMPLE 9: Find the circle of curvature of the graph of $f(x) = x^2$ at $(0, 0)$.

Use a graphing utility to graph the function along with the circle of curvature to geometrically check your answer.

Ans: $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$.

BONUS TRACKS: TORSION

Since acceleration (and hence, the force driving the motion) is contained completely in the osculating plane, and $\hat{B}$ is the direction of the osculating plane, we investigate how the direction of the osculating plane changes by looking at the rate of change of $\hat{B}$:

$$\begin{aligned} D_t[\hat{B}(t)] &= D_t[\hat{T}(t) \times \hat{N}(t)] \\ &= \hat{T}'(t) \times \hat{N}(t) + \hat{T}(t) \times \hat{N}'(t) \end{aligned}$$

Since $\hat{N}(t)$, by definition, is parallel to $\hat{T}'(t)$, $\hat{T}'(t) \times \hat{N}(t) = \vec{0}$, so $\hat{B}'(t) = \hat{T}(t) \times \hat{N}'(t)$.

Therefore we know:

- $\hat{B}'(t)$ is orthogonal to $\hat{T}(t)$, since $\hat{B}'(t)$ is a cross product with $\hat{T}(t)$ as a factor
- $\hat{B}'(t)$ is orthogonal to $\hat{B}(t)$, since $\|\hat{B}(t)\| = 1$

Hence, $\hat{B}'(t)$ is parallel to $\hat{N}$. These relations hold for any choice of parameter, in particular for the 'natural' parameter, arc length. Hence we define the **torsion**, $\tau(s)$ as follows:

$$\frac{d\hat{B}}{ds} = -\tau(s)\hat{N}(s),$$

where the '$-$' is there as a convention only. By 'dotting' both sides with $\hat{N}(s)$, we get:

$$\frac{d\hat{B}}{ds} \cdot \hat{N}(s) = \left(-\tau(s)\hat{N}(s)\right) \cdot \hat{N}(s) = -\tau(s)\left(\hat{N}(s) \cdot \hat{N}(s)\right) = -\tau(s),$$

This leads to our 'definition' of torsion as:

$$\tau = -\frac{d\hat{B}}{ds} \cdot \hat{N}(s) = -\frac{1}{\|\vec{v}(t)\|}\frac{d\hat{B}}{dt} \cdot \hat{N}(t),$$

the latter equation coming from the chain rule.

NOTE: Torsion mesures the tendency for the osculating plane to 'twist' since:

$$\|\tau\| = \left\| -\frac{d\hat{B}}{ds} \cdot \hat{N}(s) \right\| = \left\| \frac{d\hat{B}}{ds} \right\| \left\| \hat{N} \right\| \cos(0 \text{ or } \pi) = \left\| \frac{d\hat{B}}{ds} \right\|,$$

ALTERNATIVE FORMULA FOR TORSION: $\tau = \dfrac{(\vec{v} \times \vec{a}) \cdot \vec{a}\,'}{\|\vec{v} \times \vec{a}\|^2}$.

EXAMPLE 10: Find the torsion of $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$.

Ans: $\tau = \dfrac{60}{169}$

BONUS TRACKS: THE FRENET SERRET EQUATIONS:

Using the fact that $\frac{d\hat{T}}{ds} = \kappa\hat{N}$ and $\frac{d\hat{B}}{ds} = -\tau\hat{N}$, differentiate the equation $\hat{N} = \hat{B} \times \hat{T}$ to get: $\frac{d\hat{N}}{ds} = -\kappa\,\hat{T} + \tau\,\hat{B}$.

Taken together, the following are known as the 'Frenet-Serret' Equations.

$$\begin{cases} \frac{d\hat{T}}{ds} &= \kappa\,\hat{N} \\ \frac{d\hat{N}}{ds} &= -\kappa\,\hat{T} + \tau\,\hat{B} \\ \frac{d\hat{B}}{ds} &= -\tau\,\hat{N} \end{cases}$$

If interested, look up these equations and you'll see how important they are in virtually every scientific field.

HOMEWORK: Section 14.5: 11 - 75 every other odd; 74*